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Invariant Constant Multi-Valued Mapping for the Heat Conductivity Problem

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ABSTRACT

We consider a heat conductivity equation with boundary control. The problems of strong and weak invariance of constant multi-valued mapping are analyzed for this equation. The control function is given on the boundary and the problem is studied under various constraints on control. We obtain sufficient conditions of strong and weak invariance for the given multi-valued mapping.

Keywords: Invariant set, control, multi-valued mapping, systems with distributed parameters.

1. Introduction

In many problems, to keep the controlled objects in desired states is very important. One of such desired states is a multi-valued mapping (MVM) which can be strong or weak invariant depending on the realization of admissible controls. Concerning the systems with distributed parameters, important results were obtained by Feuer and Heymann (1976), Guseinov and Ushakov (1998), Rettiev (1979), Fazylov (1997), and other researchers for the problem of invariance of given sets.

In the works of Feuer and Heymann (1976), Guseinov and Ushakov (1998), Kurzhanskii and Filippova (1987), Rettiev (1979), problems in systems with concentrated parameters were studied, whereas the works of Alimov (2011), Tukhtasinov et al. (2013), Tukhtasinov and Ibragimov (2011) analysis the systems with distributed parameters. In particular, the work of Kurzhanskii and Filippova (1987) is devoted to the problem of keeping the set of trajectories of a differential inclusion till a specified time in a given MVM. To do this, the set of attainability of the given system at each time and their dependence on the survival interval are analytically described. The paper of Satimov (2006) studies a conflicted situation where control functions are on the right hand side of the equation. Some partial regularity results for the entropy solution of the so-called relativistic heat equation are proved in the work of Andreu et al. (2008). The work of Petter et al. (2017) is devoted to investigation of a synthesis problem for differential and difference equations. In the paper of Fazylov (1995), a problem on the existence of the kernel of survivability is considered.

A control process described by parabolic equation is investigated by Tukhtasinov and Ibragimov (2011), where the control parameter is on the right hand side of equation in additive form and subjected to integral constraint. Some conditions for constant parameters were obtained in that paper to guarantee strong or weak invariance of the set of interval type.

In the work of Tukhtasinov et al. (2013), similar results were obtained for the problems formulated in the work of Tukhtasinov and Ibragimov (2011) but in contrast to the work of Tukhtasinov and Ibragimov (2011) right hand side of the equation is delaying state of the system.

Interesting applied control problems on heat distribution in a volume by boundary convector radiator are studied by Alimov (2011). It should be noted that the Green function was used in that paper to solve the problem and the result differs from the real solution to within any $\varepsilon > 0$ by increasing the number of Fourier coefficients.

In the present paper, the problems of strong and weak invariance of an MVM, which is a fixed interval of the real line, are studied with respect to systems in distributed parameters. It is important to note that the control parameter is given on the boundary, that is, a boundary control problem is considered. It is crucial that the set of eigenfunctions of corresponding elliptic operator is complete in the space $L_2(\Omega)$, but is not so in the space $L_2(\partial\Omega)$. This considerably complicates the application of the method of separation of variables, which is the main method of study of present work. To overcome this situation, we propose to consider a "truncated control", namely, to consider the problem in subspaces of arbitrary dimension of space $L_2(\Omega)$. In the work of Alimov (2011), the problem was considered under geometric constraint on control. However, in the present work, the problem is studied under various constraints on the control. We obtain sufficient conditions of strong and weak invariance of a given multi-valued mapping on a given fixed interval. We consider various norms on control as well as on the state of the system.

2. Statement of problem

Let

$$Az = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial z}{\partial x_j}\Big),\ x \in \Omega,$$

and

$$Pz = \frac{\partial z}{\partial \nu} + h(x)z,\ x \in \partial\Omega,$$

with $a_{ij}(x) = a_{ji}(x) \in C^1(\Omega),\ \ i,j = 1,...,n,\ \ \Omega$ is a bounded set in $\mathbb{R}^n$ with piecewise smooth boundary, A is elliptic differential operator, that is, for some positive number γ,

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq \gamma \sum_{i=1}^{n} \zeta_i^2,$$

for all $x \in \overline{\Omega}$ and real numbers $\zeta_1,...,\zeta_n,\ \sum\limits_{i=1}^{n}\zeta_i^2 \neq 0,\ h(x)$ is a given positive continuous function, $\dfrac{\partial z}{\partial \nu}$ is derivative along the outer normal ν to the boundary of the set Ω at the point $x \in \partial\Omega$.

Consider the following heat exchange control problem (Alimov (2011), Egorov (1978))

$$\frac{\partial z(t,x)}{\partial t} = Az(t,x), \quad 0 < t \leq \mathrm{T},\ x \in \Omega, \tag{1}$$

with boundary condition

$$Pz(t,x) = u(t,x), \quad 0 \le t \le \mathrm{T}, \quad x \in \partial\Omega, \tag{2}$$

and initial condition

$$z(0,x) = z^0(x), \quad x \in \Omega, \tag{3}$$

where $z(t,x)$ is an unknown function, T is an arbitrary positive number, $z^0(\cdot) \in L_2(\Omega)$ is initial function. Controls are assumed to be measurable functions $u(\cdot,\cdot) \in L_2(S_\mathrm{T})$, where $S_\mathrm{T} = \{(t,x) \mid \ t \in [0,\mathrm{T}], \ x \in \partial\Omega\}$.

It was proved (Ladyzhenskaja, 1973, Ch. III, § 4, Sec. 2, Theorem 4.1) that for any $u(\cdot,\cdot) \in L_2(S_\mathrm{T})$ and $z^0(\cdot) \in L_2(\Omega)$, problem (1)–(3) has a unique solution $z(t,x)$ in the Hilbert space $W_2^{1,0}(Q_\mathrm{T})$ which consisted of elements of the space $L_2(Q_\mathrm{T})$ having square summable generalized derivatives $z_{x_i}, \ i = 1, ..., n$, over Q_T, where $Q_\mathrm{T} = \{(t,x) \mid t \in (0,\mathrm{T}), \ x \in \Omega\}$.

Since the elliptic operator A with boundary condition $Pz(t,x) = 0$, $0 \le t \le \mathrm{T}$, $x \in \partial\Omega$, has a discrete specter, that is, it has eigenvalues λ_k that satisfy the conditions $0 < \lambda_1 \le \lambda_2 \le ...$, and $\lambda_k \to +\infty$ as $k \to \infty$, and eigenfunctions $\varphi_k(x)$, $x \in \Omega$, that forms a complete orthonormal system in $L_2(\Omega)$ (Ladyzhenskaja et al., 1967, Ch. III, § 6).

We find the solution of problem (1)–(3) by the Fourier method. If $z_l(\cdot)$ denote the coefficients of Fourier of the function $z(\cdot,\cdot)$ with respect to the system $\{\varphi_l\}$, then the solution of problem (1)–(3) can be represented as follows:

$$z(t,x) = \sum_{l=1}^{\infty} z_l(t)\varphi_l(x), \ 0 \le t \le \mathrm{T}, \ x \in \Omega,$$

where

$$z_l(t) = z_l^0 e^{-\lambda_l t} + \int_0^t \int_{\partial\Omega} u(\tau,s)\varphi_l(s) \ e^{-\lambda_l(t-\tau)} ds \ d\tau, \tag{4}$$

and z_l^0 are the Fourier coefficients of the initial function $z^0(\cdot)$, that is,

$$z_l^0 = \int_\Omega z^0(s)\varphi_l(s)ds.$$

Next, let U be the set of control functions which will be specified later by a number $\rho > 0$. Set (Alimov (2011))

$$H_m = \left\{ z \in L_2(\Omega) \mid \ z = \sum_{l=1}^{m} \alpha_l \varphi_l(x), \ \alpha_l \in \mathbb{R} \right\}.$$

Let $S_m : L_2(\Omega) \to L_2(\Omega)$ be an orthogonal project mapping on H_m defined by

$$S_m z(t,x) = \sum_{l=1}^{m} z_l(t)\varphi_l(x).$$

Definition 2.1. *A MVM $F : [0,T] \to 2^{\mathbb{R}}$, where $\mathbb{R} = (-\infty, \infty)$, is called strong invariant with respect to problem (IWRP) (1)–(3), if for any $u(\cdot,\cdot) \in U$ and $z^0(\cdot) \in L_2(\Omega)$ with $\langle S_m z^0(\cdot)\rangle_{L_2(\Omega)} \in F(0)$, the inclusion $\langle S_m z(t,\cdot)\rangle \in F(t)$ holds for all t, $0 < t \leq T$, where $\langle\cdot\rangle$ denotes the norm, and $z(\cdot,\cdot)$ is corresponding solution of problem (1)–(3) (Tukhtasinov et al. (2013), Tukhtasinov and Ibragimov (2011)).*

Definition 2.2. *A MVM $F : [0,T] \to 2^{\mathbb{R}}$ is called weak IWRP (1)–(3), if for any $z^0(\cdot) \in L_2(\Omega)$ with $\langle S_m z^0(\cdot)\rangle \in F(0)$ there exists a control $u(\cdot,\cdot) \in U$ for which $\langle S_m z(t,\cdot)\rangle \in F(t)$ for all t, $0 < t \leq T$, where $\langle\cdot\rangle$ denotes the norm, and $z(\cdot,\cdot)$ is corresponding solution of problem (1)–(3).*

In the present paper, we study weak and strong invariance of constant MVMs of the form $F(t) = [0,b]$, $0 \leq t \leq \mathrm{T}$, for a positive number b.

3. Sufficient conditions of strong and weak invariance

In this section we obtain some relations between the parameters T, b, ρ, and λ_i under which MVM $F(t)$, $t \in [0,\mathrm{T}]$, is strong or weak IWRP (1)–(3). Let

$$\begin{aligned}
\|S_m z(t,\cdot)\| &= \left(\int_\Omega |S_m z(t,s)|^2 ds\right)^{1/2} = \left(\sum_{l=1}^{m} z_l^2(t)\right)^{1/2}, \; 0 \leq t \leq \mathrm{T},\\
\|S_m z(\cdot,\cdot)\| &= \sqrt{\int_0^{\mathrm{T}} \|S_m z(t,\cdot)\|^2 dt} = \sqrt{\sum_{l=1}^{m} \int_0^{\mathrm{T}} z_l^2(t) dt},
\end{aligned}$$

$$U_1 = \left\{ u(\cdot,\cdot) \mid \sum_{l=1}^{m} \left(\int_{\partial\Omega} u(t,s)\varphi_l(s)ds \right)^2 \leq \rho^2,\ t \in [0,\mathrm{T}] \right\},$$
$$U_2 = \left\{ u(\cdot,\cdot) \mid \sum_{l=1}^{m} \int_0^{\mathrm{T}} \left(\int_{\partial\Omega} u(\tau,s)\varphi_l(s)ds \right)^2 d\tau \leq \rho^2 \right\}.$$

We consider four cases and for each case give sufficient conditions of strong and weak invariance.

3.1 The case of norm $\|S_m z(t,\cdot)\|$ and the set of controls U_1

Let the norm is defined by $\langle S_m z(t,\cdot)\rangle = \|S_m z(t,\cdot)\|$, $0 \leq t \leq \mathrm{T}$, and control set be $U = U_1$. We prove the following statement.

Theorem 3.1. *If*

$$\rho \leq \lambda_1 b, \tag{5}$$

then MVM $F(t)$, $t \in [0,T]$, *is strong IWRP (1)–(3).*

Proof. Let $\rho \leq \lambda_1 b$. Show strong invariance of MVM $F(t)$, $0 \leq t \leq \mathrm{T}$, with respect to problem (1)–(3). Indeed, for any $S_m z(0,\cdot)$ and $u(t,\cdot)$ with

$$\|S_m z(0,\cdot)\| = \left(\sum_{l=1}^{m} |z_l^0|^2 \right)^{1/2} \leq b, \quad \|u(t,\cdot)\| \leq \rho,$$

we obtain from (4) that

$$\begin{aligned} \|S_m z(t,\cdot)\|^2 &= \int_\Omega \left| S_m z(t,x) \right|^2 dx = \sum_{l=1}^{m} z_l^2(t) \\ &= \sum_{l=1}^{m} \left(z_l^0 e^{-\lambda_l t} + \int_0^t e^{-\lambda_l (t-\tau)} \int_{\partial\Omega} u(\tau,s)\varphi_l(s)dsd\tau \right)^2 \\ &= \sum_{l=1}^{m} \Bigg(|z_l^0|^2 e^{-2\lambda_l t} + 2 z_l^0 e^{-\lambda_l t} \int_0^t e^{-\lambda_l (t-\tau)} \int_{\partial\Omega} u(\tau,s)\varphi_l(s)dsd\tau \\ &\quad + \Big(\int_0^t e^{-\lambda_l (t-\tau)} \int_{\partial\Omega} u(\tau,s)\varphi_l(s)dsd\tau \Big)^2 \Bigg). \end{aligned} \tag{6}$$

Since $\lambda_l \geq \lambda_1$, the right hand side of (6) can be estimated from above as follows

$$\begin{aligned}\sum_{l=1}^{m} |z_l^0|^2 e^{-2\lambda_1 t} \quad &+ \quad 2e^{-\lambda_1 t} \int_0^t e^{-\lambda_1 (t-\tau)} \left(\sum_{l=1}^{m} |z_l^0| \left| \int_{\partial\Omega} u(\tau, s)\varphi_l(s) ds \right| \right) d\tau \\ &+ \quad \sum_{l=1}^{m} \left(\int_0^t e^{-\lambda_l (t-\tau)} \int_{\partial\Omega} u(\tau, s)\varphi_l(s) ds d\tau \right)^2 . \end{aligned} \tag{7}$$

Estimate the third term of (7). Since $\lambda_l \geq \lambda_1$, then using the Cauchy-Schwartz inequality we obtain for any function $u(\cdot, \cdot) \in U_1$ that

$$\begin{aligned} &\sum_{l=1}^{m} \left(\int_0^t \int_{\partial\Omega} e^{-\lambda_l (t-\tau)} u(\tau, s)\varphi_l(s) ds d\tau \right)^2 \\ &= \sum_{l=1}^{m} \left(\int_0^t e^{-\frac{\lambda_l}{2}(t-\tau)} \int_{\partial\Omega} e^{-\frac{\lambda_l}{2}(t-\tau)} u(\tau, s)\varphi_l(s) ds d\tau \right)^2 \\ &\leq \sum_{l=1}^{m} \left(\int_0^t e^{-\lambda_l (t-\tau)} d\tau \cdot \int_0^t e^{-\lambda_l (t-\tau)} \Big(\int_{\partial\Omega} u(\tau, s)\varphi_l(s) ds \Big)^2 d\tau \right) \\ &\leq \left(\int_0^t e^{-\lambda_1 (t-\tau)} d\tau \right)^2 \rho^2 = \left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1} \right)^2 \rho^2 . \end{aligned} \tag{8}$$

Applying the Cauchy inequality to the second term of (7) and using (6) and (8), we get

$$\|S_m z(t, \cdot)\|^2 \leq b^2 e^{-2\lambda_1 t} + 2\, b\rho e^{-\lambda_1 t} \int_0^t e^{-\lambda_1 (t-\tau)} d\tau + \left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1} \right)^2 \rho^2 .$$

Therefore,

$$\|S_m z(t, \cdot)\| \leq b e^{-\lambda_1 t} + \rho \frac{1 - e^{-\lambda_1 t}}{\lambda_1} . \tag{9}$$

Inequalities (5) and (9) imply that

$$\|S_m z(t, \cdot)\| \leq b.$$

This means that MVM $F(t),\ 0 \leq t \leq \mathrm{T}$, is strong IWRP (1)–(3). Proof of Theorem 3.1 is complete. □

Remark 3.1. *It can be shown that MVM $F(t),\ 0 \leq t \leq T$, is weak IWRP (1)–(3).*

To verify this we have to show that, for any $z^0(\cdot) \in L_2(\Omega)$ with $\|S_m z^0(\cdot)\| \leq b$, one can choose a control $u(\cdot,\cdot) \in U$ so that MVM $F(t),\ 0 \leq t \leq \mathrm{T}$, is weak IWRP (1)–(3).

Indeed, let $u(t,x) = 0,\ t \geq 0,\ x \in \partial\Omega$. Then for any $z^0(\cdot) \in L_2(\Omega)$ with $\|S_m z^0(\cdot)\| \leq b$, we have

$$\begin{aligned}
\|S_m z(t,\cdot)\|^2 &= \sum_{l=1}^{m}\left(z_l^0 e^{-\lambda_l t} + \int_0^t e^{-\lambda_l(t-\tau)} \int_{\partial\Omega} u(\tau,s)\varphi_l(s)\,ds\,d\tau\right)^2 \\
&= \sum_{l=1}^{m}\left(z_l^0 e^{-\lambda_l t}\right)^2 \leq \sum_{l=1}^{m}|z_l^0|^2 \leq b^2,
\end{aligned}$$

which is the desired conclusion.

3.2 The case of norm $\|S_m z(\cdot,\cdot)\|$ and the set of controls U_2

Let $\langle S_m z(\cdot,\cdot)\rangle = \|S_m z(\cdot,\cdot)\|$, and $U = U_2$. Prove the following statement of strong invariance.

Theorem 3.2. *If $\rho \leq b\sqrt{2\lambda_1(1-T)/T}$, then MVM $F(t),\ 0 \leq t \leq T$, is strong IWRP (1)–(3).*

Proof. Let $\rho \leq b\sqrt{2\lambda_1(1-\mathrm{T})/\mathrm{T}}$. Assume that $z^0(\cdot)$ is an element of the space $L_2(\Omega)$ which satisfies the condition $\|S_m z^0(\cdot)\| \in F(0)$, that is, $\|S_m z^0(\cdot)\| \leq b$. Let $u(\cdot,\cdot)$ be any admissible control, that is, $u(\cdot,\cdot) \in U_2$. Derive an estimate for $\|S_m z(t,\cdot)\|$.

Using the inequalities $\lambda_l \geq \lambda_1$ and the Cauchy-Schwartz inequality, we can estimate the right hand side of the inequality (6) from above as follows

$$\sum_{l=1}^{m}\Big(|z_l^0|^2e^{-2\lambda_1 t}+2e^{-\lambda_1 t}|z_l^0|\sqrt{\int\limits_0^t e^{-2\lambda_1(t-\tau)}d\tau}\sqrt{\int\limits_0^t\Big(\int\limits_{\partial\Omega}u(\tau,s)\varphi_l(s)\,ds\Big)^2d\tau}$$

$$+\int\limits_0^t e^{-2\lambda_1(t-\tau)}d\tau\int\limits_0^t\Big(\int\limits_{\partial\Omega}u(\tau,s)\varphi_l(s)\,ds\Big)^2d\tau\Big)$$

$$\le b^2e^{-2\lambda_1 t}+2\,b\,\rho\,e^{-\lambda_1 t}\sqrt{\int\limits_0^t e^{-2\lambda_1(t-\tau)}d\tau}+\rho^2\int\limits_0^t e^{-2\lambda_1(t-\tau)}d\tau$$

$$=\left(be^{-\lambda_1 t}+\rho\sqrt{\int\limits_0^t e^{-2\lambda_1(t-\tau)}d\tau}\right)^2$$

Therefore,

$$\|S_m z(t,\cdot)\|\le b\,e^{-\lambda_1 t}+\frac{\rho}{\sqrt{2\lambda_1}}\sqrt{1-e^{-2\lambda_1 t}}. \tag{10}$$

To estimate the right hand side of (10), denote

$$f(t)=b\,e^{-\lambda_1 t}+\frac{\rho}{\sqrt{2\lambda_1}}\sqrt{1-e^{-2\lambda_1 t}},\ t\ge 0.$$

Show that for any positive parameters b and ρ,

$$\max_{t\ge 0}f(t)=f(t_0)=\sqrt{b^2+\frac{\rho^2}{2\lambda_1}},\quad t_0=\frac{1}{2\lambda_1}\ln\Big(1+\frac{\rho^2}{2\lambda_1 b^2}\Big).$$

Indeed, we have

$$f'(t)=\frac{\lambda_1 e^{-\lambda_1 t}}{\sqrt{2\lambda_1}\sqrt{1-e^{-2\lambda_1 t}}}\Big(\rho\,e^{-\lambda_1 t}-b\sqrt{2\lambda_1}\sqrt{1-e^{-2\lambda_1 t}}\Big).$$

We can see that

$$\begin{aligned}f'(t)&\ \ge\ f'(t_0)=0,\quad 0<t\le t_0,\\ f'(t)&\ <\ f'(t_0)=0,\quad t_0<t.\end{aligned}$$

Hence,

$$\max_{t>0} f(t) = f(t_0) = \sqrt{b^2 + \frac{\rho^2}{2\lambda_1}}.$$

Thus,

$$\|S_m z(t,\cdot)\| \leq \sqrt{b^2 + \frac{\rho^2}{2\lambda_1}}. \tag{11}$$

It follows from (10) and (11) that

$$\begin{aligned} \|S_m z(\cdot,\cdot)\|^2 &= \int_0^{\mathrm{T}} \|S_m z(t,\cdot)\|^2 dt \leq \int_0^{\mathrm{T}} |f(t)|^2 dt \\ &\leq \int_0^{\mathrm{T}} |f(t_0)|^2 dt = \Big(b^2 + \frac{\rho^2}{2\lambda_1}\Big)\mathrm{T} \leq b^2. \end{aligned}$$

Therefore, $\|S_m z(\cdot,\cdot)\| \in F(t)$ for all $0 \leq t \leq \mathrm{T}$, that is, $F(t)$ is strong IWRP (1)–(3), which proves the theorem. □

Theorem 3.3. *If* $2\lambda_1 \geq 1$, *then MVM* $F(t)$, $0 \leq t \leq T$, *is weak IWRP (1)–(3).*

Proof. Let $2\lambda_1 \geq 1$. Show that MVM $F(t)$, $0 \leq t \leq \mathrm{T}$, is weak invariant. Indeed, for any $z^0(\cdot) \in L_2(\Omega)$ with $\|S_m z^0(\cdot)\| \in F(0)$, we choose the control $u(t,x) = 0$, $t \in [0,\mathrm{T}]$, $x \in \partial\Omega$, and so $\int_{\partial\Omega} u(t,s)\varphi_l(s)ds = 0$, $l = 1,2,....$ Then,

$$\begin{aligned} \|S_m z(\cdot,\cdot)\|^2 &= \int_0^{\mathrm{T}} \|S_m z(t,\cdot)\|^2 dt = \sum_{l=1}^{m} \int_0^{\mathrm{T}} \Big(z_l^0 e^{-\lambda_l t}\Big)^2 dt \\ &\leq b^2 \frac{1 - e^{-2\lambda_1 \mathrm{T}}}{2\lambda_1} \leq \frac{b^2}{2\lambda_1} \leq b^2. \end{aligned}$$

This implies that $\|S_m z(\cdot,\cdot)\| \in F(t)$ for all $0 \leq t \leq \mathrm{T}$, and hence $F(t), 0 \leq t \leq \mathrm{T}$, is weak IWRP (1)–(3). This is our assertion. □

3.3 The case of norm $\|S_m z(\cdot,\cdot)\|$ and the set of controls U_1

Let $\langle S_m z(\cdot,\cdot)\rangle = \|S_m z(\cdot,\cdot)\|$, and $U = U_1$. Prove theorems on strong and weak invariance.

Theorem 3.4. *If either* $\rho \leq \lambda_1 b, \ T \leq 1,$ *or*

$$1 < \rho/(\lambda_1 b) \leq (1 - \sqrt{T} e^{-\lambda_1 T})/(\sqrt{T}(1 - e^{-\lambda_1 T})), \tag{12}$$

then MVM $F(t), \ 0 \leq t \leq T,$ *is strong IWRP (1)–(3).*

Proof. Establish that, for any $z^0(\cdot) \in L_2(\Omega)$ with $\|S_m z^0(\cdot)\| \leq b$ and $\|u(\cdot,\cdot)\| \leq \rho$, the inclusion $\|S_m z(\cdot,\cdot)\| \in F(t)$, $0 \leq t \leq \mathrm{T}$ holds. Consider the function

$$g(t) = b \ e^{-\lambda_1 t} + \rho \ \frac{1 - e^{-\lambda_1 t}}{\lambda_1}, \ t \geq 0. \tag{13}$$

It is not difficult to verify that

$$\max_{0 \leq t \leq \mathrm{T}} g(t) = \begin{cases} g(0) = b, & \rho \leq \lambda_1 b, \\ g(\mathrm{T}) = \frac{\rho}{\lambda_1} + \left(b - \frac{\rho}{\lambda_1}\right) e^{-\lambda_1 \mathrm{T}}, & \rho > \lambda_1 b. \end{cases} \tag{14}$$

Observe that (9), (13), and (14) imply the following relations

$$\|S_m z(\cdot,\cdot)\|^2 = \int_0^{\mathrm{T}} \|S_m z(t,\cdot)\|^2 dt \leq \int_0^{\mathrm{T}} g^2(t) dt \leq \left(\max_{0 \leq t \leq \mathrm{T}} g(t)\right)^2 \mathrm{T}.$$

Let $\rho \leq \lambda_1 b, \ \mathrm{T} \leq 1$. Then using (14) we obtain

$$\|S_m z(\cdot,\cdot)\| \leq \max_{0 \leq t \leq \mathrm{T}} g(t) \sqrt{\mathrm{T}} = g(0)\sqrt{\mathrm{T}} = b\sqrt{\mathrm{T}} \leq b.$$

Let now (12) be satisfied. Then

$$\begin{aligned} \|S_m z(\cdot,\cdot)\| &\leq \max_{0 \leq t \leq \mathrm{T}} g(t)\sqrt{\mathrm{T}} = g(\mathrm{T})\sqrt{\mathrm{T}} \\ &= \left[b \, e^{-\lambda_1 \mathrm{T}} + \frac{\rho}{\lambda_1}(1 - e^{-\lambda_1 \mathrm{T}})\right]\sqrt{\mathrm{T}} \leq b. \end{aligned}$$

Thus, $F(t), \ t \in [0, \mathrm{T}]$, is strong invariant and the theorem follows.

□

Theorem 3.5. *If* $2\lambda_1 \geq 1$, *then MVM* $F(t), \ 0 \leq t \leq T,$ *is weak IWRP (1)–(3).*

The theorem can be proved similar to Theorem 3.3.

3.4 The case of norm $\|S_m z(t,\cdot)\|$ and the set of controls U_2

Let $\langle S_m z(t,\cdot)\rangle = \|S_m z(t,\cdot)\|$, and $U = U_2$. Denote $\xi_l(t) = \sqrt{\int\limits_0^t e^{2\lambda_l \tau} d\tau}$.

Prove the following statement.

Theorem 3.6. *Let for a number $t^* \in (0, T)$ that satisfies the inequality*

$$\rho > \lambda_m(b\xi_m(t^*) + \rho\xi_m^2(t^*)) \tag{15}$$

there exist a control $u(\cdot,\cdot) \in U$ such that $\int\limits_{\partial\Omega} u(\tau,s)\varphi_i(s)ds = 0,\ 0 \le \tau \le T,\ i \ne l$,

$$\int\limits_{\partial\Omega} u(\tau,s)\varphi_l(s)ds = \frac{\rho}{\xi_l(t^*)}e^{\lambda_l \tau},\ 0 \le \tau \le t^*,\quad \int\limits_{\partial\Omega} u(\tau,s)\varphi_l(s)ds = 0,\ t^* < \tau \le T,$$

at some $l \in \{1, ..., m\}$. Then MVM $F(t)$, $t \in [0, T]$, is not strong IWRP (1)–(3) on $[0, T]$.

Proof. The fact that values of the control on some small time interval can be made as big as we wish plays key role in the proof of the theorem. To construct such a control, take $z_l^0 = b,\ z_i^0 = 0,\ i \ne l$. Then, using (4) we obtain

$$S_m z(t) = \left(e^{-\lambda_l t} z_l^0 + \int\limits_0^t e^{-\lambda_l (t-\tau)} \frac{e^{\lambda_l \tau}}{\xi_l(t^*)} \rho d\tau \right) \varphi_l,\ 0 \le t \le t^*.$$

Hence,

$$||S_m z(t)|| = e^{-\lambda_l t}\left(b + \rho\frac{\xi_l^2(t)}{\xi_l(t^*)}\right). \tag{16}$$

Letting $\chi(t) = ||S_m z(t)||$ we can see from (16) that $\chi(0) = b$ and

$$\begin{aligned} \chi'(t) &= e^{-\lambda_l t}\left(-\lambda_l b - \frac{\lambda_l \rho}{\xi_l(t^*)}\xi_l^2(t) + \rho\frac{e^{2\lambda_l t}}{\xi_l(t^*)}\right) \\ &\ge e^{-\lambda_l t}\left(-\lambda_l b - \lambda_l \rho \xi_l(t^*) + \rho\frac{e^{2\lambda_l t}}{\xi_l(t^*)}\right). \end{aligned}$$

Since $\lambda_l \le \lambda_m$, then in view of (15) we get $\chi'(t) > 0$ for all $t \in [0, t^*]$ and hence $\chi(t) > b$, $t \in [0, t^*]$. Therefore we can conclude that $||S_m z(t|| \in F(t)$ not for all $t \ge 0$. The proof of the theorem is complete. □

Remark 3.2. *It can be shown that MVM $F(t)$, $0 \le t \le T$, is weak IWRP (1)–(3).*

4. Conclusion

We have studied the problems of strong and weak invariance of an MVM that is a fixed interval of the real line. The boundary control problems have been studied with respect to systems in distributed parameters under integral and geometric constraints on control function. The circumstance that the set of eigenfunctions of elliptic operator in the equation is complete in the space $L_2(\Omega)$, but is not so in the space $L_2(\partial\Omega)$, considerably complicates the application of the method of separation of variables. To overcome this situation, we have proposed to consider a "truncated control". Sufficient conditions of strong and weak invariance of a given fixed interval have been obtained.

To maintain the amount of heat in a volume within a certain range is of great practical importance. To solve such problems by means of control of boundary ingress of air heat, the method proposed in the present paper can be used.

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